



The scalarised Schwarzschild–NUT spacetime

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ABSTRACT

It has recently been suggested that vacuum black holes of General Relativity (GR) can become spontaneously scalarised when appropriate non-minimal couplings to curvature invariants are considered. These models circumvent the standard black hole no scalar hair theorems of GR, allowing both the standard GR solutions and new scalarised (a.k.a. hairy) solutions, which in some cases are thermodynamically preferred. Up to now, however, only (static and spherically symmetric) scalarised Schwarzschild solutions have been considered. It would be desirable to take into account the effect of rotation; however, the higher curvature invariants introduce a considerable challenge in obtaining the corresponding scalarised rotating black holes. As a toy model for rotation, we present here the scalarised generalisation of the Schwarzschild–NUT solution, taking either the Gauss–Bonnet (GB) or the Chern–Simons (CS) curvature invariant. The NUT charge n endows spacetime with “rotation”, but the angular dependence of the corresponding scalarised solutions factorises, leading to a considerable technical simplification. For GB, but not for CS, scalarisation occurs for $n = 0$. This basic difference leads to a distinct space of solutions in the CS case, in particular exhibiting a double branch structure. In the GB case, increasing the horizon area demands a stronger non-minimal coupling for scalarisation; in the CS case, due to the double branch structure, both this and the opposite trend are found. We briefly comment also on the scalarised Reissner–Nordström–NUT solutions.

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1. Introduction

It has long been known that violations of the strong equivalence principle, via the inclusion of non-minimal couplings, can lead to asymptotically flat black hole (BH) scalar “hair” – see [1–7] and [8–10] for recent reviews. More recently [11–13], it has been appreciated that for a wide class of non-minimal couplings, the model accommodates *both* scalarised BHs and the standard vacuum General Relativity (GR) solutions (see also [14–16]). This led to the conjecture that a phenomenon of “spontaneous scalarisation” occurs in these models [11,12], akin to the spontaneous scalarisation of neutron stars first discussed in [17], within scalar tensor theories, but with the key difference that the phenomenon is triggered by strong gravity rather than by matter. Indeed, for some choices of the function defining the non-minimal coupling, scalarised BHs are thermodynamically preferred over the GR solu-

tions and linearly stable [18], suggesting such spontaneous scalarisation occurs dynamically.

Black hole spontaneous scalarisation was indeed confirmed to occur dynamically in a different class of model [19], albeit within the same universality class related to non-minimal couplings. In this case, the scalar field non-minimally couples to the Maxwell invariant, rather than to higher curvature invariants, leading to a phenomenon of spontaneous scalarisation for electro-vacuum GR BHs. The technical simplification resulting from the absence of the higher curvature terms, allowed, besides showing dynamically the occurrence of spontaneous scalarisation, to explore the space of non-spherical static solutions that the model contains.¹

In all cases mentioned above, only static scalarised solutions have been considered, which connect to the Schwarzschild (or Reissner–Nordström) solution in the vanishing scalar field limit. It would be of interest to include rotation in this analysis, by considering the scalarisation of the Kerr solution. This is, how-

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¹ Charged scalarised BHs were also recently considered in models with a scalar field non-minimally coupled to higher curvature corrections [20].

ever, considerably more challenging than the spherically symmetric cases, due to the combination of the higher curvature corrections and the higher co-dimensionality of the problem, due to the smaller isometry group.

In this paper we analyse the scalarisation of a different generalisation of the Schwarzschild solution, the Taub–Newman–Tamburino–Unti solution (Taub–NUT) [21,22]. This solution of the Einstein vacuum field equations can be regarded as the Schwarzschild solution with a NUT “charge” n (hereafter referred to as the Schwarzschild–NUT solution) which endows the spacetime with rotation in the sense of promoting dragging of inertial frames. It is often described as a ‘gravitational dyon’ with both ordinary and magnetic mass. The NUT charge n plays a dual role to ordinary ADM mass M , in much the same way that electric and magnetic charges are dual within Maxwell’s electromagnetism [23]. The Schwarzschild–NUT solution is not asymptotically flat in the usual sense, although it does obey the required fall-off conditions [24]; for scalarisation, nonetheless, which is mostly supported in the neighbourhood of the horizon, the non-standard asymptotics are of lesser importance. In fact, the existence of scalarised solutions with these asymptotics supports the universality of the scalarisation phenomenon. The Schwarzschild–NUT solution is a case study in GR, with a number of other unusual properties [24], and its Euclideanised version has been suggested to play a role in the context of quantum gravity [25].

Here, we shall exhibit some basic properties of scalarised Schwarzschild–NUT solution. This scalarisation will have a geometric origin, occurring due to a non-minimal coupling of a scalar field and a higher curvature invariant. We shall consider two cases. The first case is the Einstein–Gauss–Bonnet–scalar (EGBs) model, which uses the four dimensional Euler density (Gauss–Bonnet invariant). The corresponding solutions described herein generalise for a non-zero NUT parameter the recently discovered scalarised Schwarzschild BHs in [11–13]. The Kerr-like form of the Schwarzschild–NUT spacetime also allows, moreover, considering a second case, the Einstein–Chern–Simons–scalar (ECSs) model, wherein the geometric scalarisation occurs due to a coupling between the scalar field and the Pontryagin density.

This paper is structured as follows. In Section 2 we review the basic framework, including the field equations and ansatz. The main results of this work are given in Section 3. First, we present a test field analysis, wherein the backreaction of the scalar field on the metric is neglected. The corresponding solutions – *scalar clouds* – occur at branching points of the Schwarzschild–NUT solution, effectively requiring a quantisation of the parameters of this solution. Then the non-perturbative solutions are also studied. In particular, we exhibit their domain of existence in terms of the relevant parameters. In Section 4 we summarise our results and provide some further remarks together with possible avenues for future research.

2. The model

2.1. The action and field equations

We consider the following action functional

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + f(\phi) \mathcal{I}(g) \right], \quad (2.1)$$

that describes a generalised gravitational theory containing the Ricci scalar curvature R and a massless real scalar field ϕ which is non-minimally coupled to the spacetime geometry, via a coupling function $f(\phi)$ and a source term $\mathcal{I}(g)$, which is built in terms of the metric tensor and its derivatives only.

We shall consider two cases for $\mathcal{I}(g)$, corresponding to two different models²:

- i) the Einstein–Gauss–Bonnet–scalar (EGBs) model: $\mathcal{I} = \mathcal{L}_{GB} \equiv R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}$,
- ii) the Einstein–Chern–Simons–scalar (ECSs) model: $\mathcal{I} = \mathcal{L}_{CS} \equiv R\tilde{R} = {}^*R^a{}_b{}^{cd}R^b{}_{acd}$.

Both \mathcal{L}_{GB} and \mathcal{L}_{CS} are topological (total derivatives) in four dimensions. Due to the coupling to the scalar field ϕ specified by $f(\phi)$, however, they become dynamical, contributing to the equations of motion.

The field equations obtained by varying (2.1) with respect to ϕ and g_{ab} are

$$\nabla^2 \phi + \frac{df(\phi)}{d\phi} \mathcal{I} = 0, \quad (2.3)$$

for the scalar field, and

$$G_{ab} = \frac{1}{2} T_{ab}^{(\text{eff})}, \quad (2.4)$$

for the gravitational field. We have chosen to present (2.4) in a GR-like form, where G_{ab} is the Einstein tensor and $T_{ab}^{(\text{eff})}$ is the effective energy momentum tensor,

$$T_{ab}^{(\text{eff})} \equiv T_{ab}^{(\phi)} + T_{ab}^{(\text{grav})}, \quad (2.5)$$

which is a combination of the scalar field energy–momentum tensor,

$$T_{ab}^{(\phi)} \equiv (\nabla_a \phi)(\nabla_b \phi) - \frac{1}{2} g_{ab} (\nabla_c \phi)(\nabla^c \phi), \quad (2.6)$$

and a supplementary contribution which depends on the explicit form of \mathcal{I} . For the cases herein we have

$$T_{ab}^{(\text{grav})} = - (g_{ak}g_{bj} + g_{bk}g_{aj}) \eta^{khcd} \eta^{ijef} R_{cdef} \nabla_h \nabla_i f(\phi), \quad (2.7)$$

for a GB source term, and

$$T_{ab}^{(\text{grav})} = -8C_{ab}, \quad \text{with} \\ C^{ab} = [\nabla_c f(\phi)] \epsilon^{cde(a} \nabla_e R^{b)}_d + [\nabla_c \nabla_d f(\phi)] {}^*R^{d(ab)c}, \quad (2.8)$$

in the CS case.

2.2. The spontaneous scalarization mechanism

The possible occurrence of a ‘spontaneous scalarisation’ in this class of models relies on two ingredients. Firstly, there must exist a *scalar-free solution*, with

$$\phi = \phi_0, \quad (2.9)$$

which is the fundamental solution of equation (2.3). This requirement implies the coupling function should satisfy the condition

$$\left. \frac{\partial f}{\partial \phi} \right|_{\phi=\phi_0} = 0. \quad (2.10)$$

This condition signifies that the usual GR solutions also solve the considered model, forming the fundamental set. We remark one

² ${}^*R^a{}_b{}^{cd}$ is the Hodge dual of the Riemann-tensor

$${}^*R^a{}_b{}^{cd} \equiv \frac{1}{2} \eta^{cdef} R^a{}_{bef}, \quad (2.2)$$

where η^{cdef} is the 4-dimensional Levi-Civita tensor, $\eta^{cdef} = \epsilon^{cdef} / \sqrt{-g}$ and ϵ^{cdef} the Levi-Civita tensor density.

can set $\phi_0 = 0$ without any loss of generality (via a field redefinition).

Secondly, the model possesses a second set of solutions, with a nontrivial scalar field – *the scalarised solutions*. In the asymptotically flat case, these solutions can be entropically preferred over the fundamental ones (i.e. they maximise the entropy for given global charges) [18,19]. Moreover, they are smoothly connected with the scalar-free solution, approaching it for $\phi = 0$.

2.3. The scalar-free Schwarzschild-NUT solution

The scalar-free solution we wish to consider, within the model (2.3)–(2.4), is the Schwarzschild-NUT solution. It has $\phi = 0$ and line element:

$$ds^2 = -N(r)\sigma^2(r)(dt + 2n \cos\theta d\varphi)^2 + \frac{dr^2}{N(r)} + g(r)(d\theta^2 + \sin^2\theta d\varphi^2), \quad (2.11)$$

with

$$g(r) = r^2 + n^2, \quad (2.12)$$

and

$$N(r) = 1 - \frac{2(Mr + n^2)}{r^2 + n^2}, \quad \sigma(r) = 1. \quad (2.13)$$

In (2.11), θ and φ are the standard angles parameterising an S^2 with the usual range. As usual, we define the NUT parameter n (with $n \geq 0$, without any loss of generality), in terms of the coefficient appearing in the differential $dt + 2n \cos\theta d\varphi$. Also, r and t are the ‘radial’ and ‘time’ coordinates.

This metric possesses a horizon located at³

$$r_h = M + \sqrt{M^2 + n^2} > 0. \quad (2.14)$$

As in the Schwarzschild limit, $N(r_h) = 0$ is only a coordinate singularity where all curvature invariants are finite. In fact, a nonsingular extension across this null surface can be found [26].

Both the Gauss-Bonnet and Pontryagin densities are non-vanishing in this background:

$$\mathcal{L}_{GB} = \frac{48M^2}{g^6} [g^2 - 16n^2r^2](r^2 - n^2) \left[1 + \frac{n}{M} \frac{(r-n)(g+4nr)}{(r+n)(g-4nr)} \right] \times \left[1 - \frac{n}{M} \frac{(r+n)(g-4nr)}{(r-n)(g+4nr)} \right], \quad (2.15)$$

and

$$\mathcal{L}_{CS} = \frac{24n}{rg^2} \left[1 - \frac{N(r^2 - 3n^2)}{g} \right] \left[1 - \frac{N(3r^2 - n^2)}{g} \right]. \quad (2.16)$$

In particular, the vanishing of \mathcal{L}_{CS} for $n = 0$, justifies the interpretation of n as a gravitomagnetic ‘mass’.

We remark that the thermodynamical description of (Lorentzian signature) solutions with NUT charge is still poorly understood. Consequently, the solutions’ properties in this respect will not be pursued herein (e.g. the usual entropy comparison between the vacuum and scalarised solutions). Still, the mass of solutions with NUT charge can be computed by employing the quasi-local formalism supplemented with the boundary counter-term method [27].

A direct computation shows that, similar to the Einstein gravity case, the mass of the solutions is identified with the constant M in the far field expansion of the metric function g_{tt} ,

$$g_{tt} = -N\sigma^2 = -1 + \frac{2M}{r} + \dots \quad (2.17)$$

2.4. Ansatz and choice of coupling

To consider the scalarised generalisations of the Schwarzschild-NUT spacetime, we consider again the metric ansatz (2.11). The metric functions $N(r)$, $\sigma(r)$ and $g(r)$ are now unknown, that shall be determined from the requirement that this metric solves the field equations (2.3)–(2.4). We have kept some metric gauge freedom in (2.11), which shall be conveniently fixed later.

The scalar field is a function of r only,

$$\phi \equiv \phi(r). \quad (2.18)$$

In this work, we shall restrict ourselves to the simplest coupling function which satisfies the condition (2.10) (with $\phi_0 = 0$):

$$f(\phi) = \alpha\phi^2, \quad (2.19)$$

with the coupling constant α an input parameter.

The equations satisfied by $N(r)$, $\sigma(r)$, $g(r)$ and $\phi(r)$ are rather involved and unenlightening; we shall not include them here. We notice, however, that they can also be derived from the following effective action⁴:

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_E + \mathcal{L}_\phi + \alpha\mathcal{L}_{\mathcal{I}\phi}, \quad (2.20)$$

where

$$\mathcal{L}_E = 2\sigma \left[1 + \left(\frac{N'}{2N} + \frac{g'}{4g} + \frac{\sigma'}{\sigma} \right) Ng' + \frac{\sigma^2 N}{g} n^2 \right],$$

$$\mathcal{L}_\phi = -\frac{1}{2} N\sigma g\phi'^2,$$

and $\mathcal{L}_{\mathcal{I}\phi} = \mathcal{L}_{GB\phi}$ or $\mathcal{L}_{\mathcal{I}\phi} = \mathcal{L}_{CS\phi}$, where

$$\mathcal{L}_{GB\phi} = 8\sigma N \left[\left(\frac{N'}{N} + \frac{2\sigma'}{\sigma} \right) \left(1 - \frac{Ng'^2}{4g} \right) + \frac{n^2 N\sigma^2}{g} \left(\frac{3N'}{N} - \frac{2g'}{g} + \frac{6\sigma'}{\sigma} \right) \right] \phi\phi',$$

and

$$\mathcal{L}_{CS\phi} = 16nN^2\sigma^2 \left[\frac{1}{4} \left(\frac{N'}{N} - \frac{g'}{g} + \frac{4\sigma'}{\sigma} \right) \left(\frac{N'}{N} - \frac{g'}{g} \right) + \frac{\sigma'^2}{\sigma^2} - \frac{1}{gN} - \frac{2n^2\sigma^2}{g^2} \right] \phi\phi'.$$

Remarkably, one can see that, due to the factorisation of the angular dependence permitted by the metric ansatz (2.11), all functions solve *second* order equations of motion *also* in ECSs case.⁵

The reduced action (2.20) makes transparent the residual scaling symmetry of the problem (λ is a nonzero constant)

$$\alpha \rightarrow \alpha\lambda^2, \quad r \rightarrow \lambda r, \quad n \rightarrow \lambda n, \quad \text{and} \quad g \rightarrow \lambda^2 g, \quad (2.21)$$

corresponding to the freedom in choosing a unit length. Therefore the solutions are characterised by dimensionless quantities such α/M^2 , Q_s/M and n/M , which shall be used below to describe them (Q_s, M shall be defined below, cf. Eq. (3.29)).

³ For $n \neq 0$, a negative value of the ‘electric’ mass M is allowed for the NUT solution. Such configurations are found for $0 < r_h < n$ and do not possess a Schwarzschild limit. As such, they are ignored in this work.

⁴ In these expressions a prime denotes a derivative w.r.t. the radial coordinate r .
⁵ Without this factorisation, the metric functions would solve third order partial differential equations, as in the case of the Kerr metric in ECSs theory, with $f(\phi) = \alpha\phi$ [28].

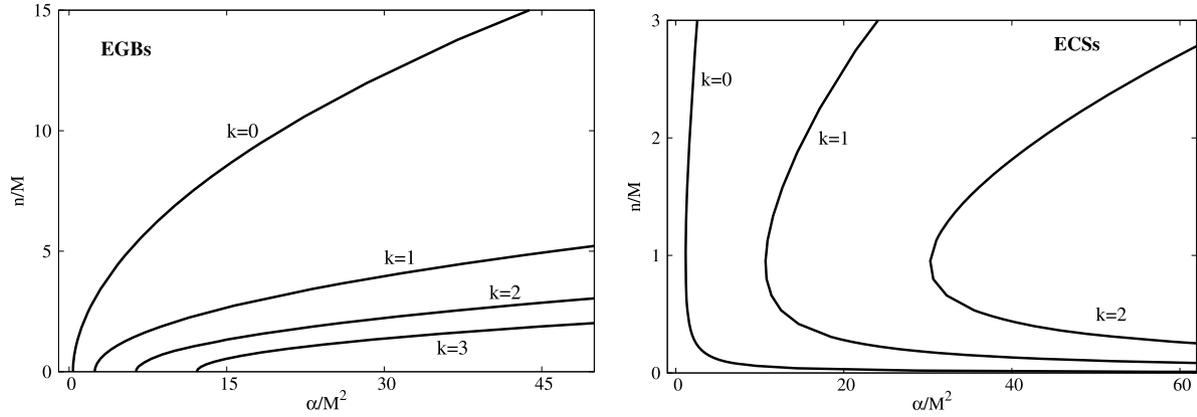


Fig. 1. The $k = 0, 1, 2$ existence lines for the EGBs (left panel) and the ECSs (right panel) models in a NUT charge vs. coupling parameter diagram.

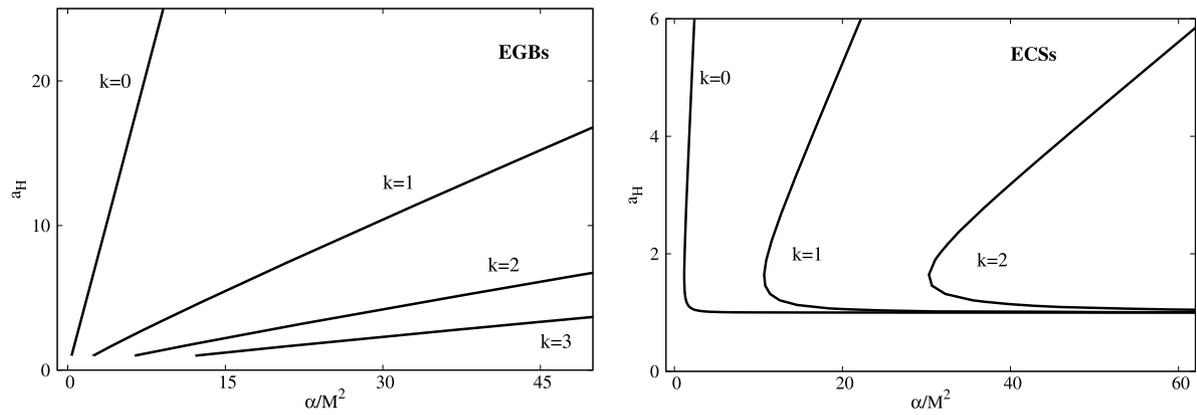


Fig. 2. The $k = 0, 1, 2$ existence lines for the EGBs (left panel) and the ECSs (right panel) models in a reduced horizon area $a_H = A_H/(16\pi M^2)$ vs. coupling parameter diagram, where $A_H = 4\pi g(r_h)$ is the area of the round- S^2 part of the line element (2.11) evaluated at the horizon.

3. Scalarised solutions

3.1. The zero-mode

We first consider the test field limit. That is, we focus on the scalar field equation (2.3) on a fixed Schwarzschild-NUT background:

$$\frac{1}{g(r)} \frac{d}{dr} [g(r)N(r)\phi'] + 2\alpha\phi\mathcal{I} = 0, \quad (3.22)$$

where \mathcal{I} is given by (2.15) or (2.16), respectively. For our purpose, solving (3.22) is an eigenvalue problem: selecting appropriate boundary conditions and fixing the values of (n, α) , selects a discrete set of BHs, as specified by the mass parameter M . These are the bifurcation points of the scalar-free solutions. The (test) scalar field profiles they support – hereafter scalar clouds – are distinguished by the number of nodes of ϕ , $k \geq 0$.

The boundary behaviours of the scalar field are regularity on the horizon and vanishing at infinity. More concretely, the approximate expression of the solution close to the horizon reads

$$\phi(r) = \phi_0 + \phi_1(r - r_h) + \mathcal{O}(r - r_h)^2, \quad (3.23)$$

with

$$\begin{aligned} \phi_1 &= \frac{24\alpha\phi_0(n^2 - r_h^2)}{r_h(r_h^2 + n^2)^2} \quad \text{for EGBs} \quad \text{and} \\ \phi_1 &= -\frac{48\alpha\phi_0 n}{(r_h^2 + n^2)^2} \quad \text{for ECSs.} \end{aligned} \quad (3.24)$$

At infinity one finds

$$\phi(r) = \frac{Q_s}{r} + \frac{Q_s(r_h^2 - n^2)}{2r_h r^2} + \dots, \quad (3.25)$$

in both cases, with Q_s a constant fixed by numerics. No exact solution appears to exist even for $n = 0$ and equation (3.22) is solved numerically. The corresponding results – the *existence lines* – for the EGBs and ECSs are shown in Fig. 1, in units given by the mass parameter M . Amongst the notable features we highlight: (i) the existence in both cases, of a minimal value of the coupling constant α below which no zero modes with a given number of nodes exists⁶; (ii) the CS model possess a double branch structure, with the existence of two different critical NUT solutions for the same (α, M) , which are distinguished by the value of the NUT charge.

In Fig. 2, the same existence lines are shown in terms of the reduced horizon area. One observes that the larger the horizon grows, the stronger the coupling has to be at the branching point, for the EGBs case. For the ECSs case, on the other hand, both this and the opposite trend can be found, corresponding to the two branches.

3.2. Non-perturbative results

3.2.1. Asymptotics and remarks on numerics

We are interested in scalarised solutions whose far field asymptotics are similar, to leading order, to those the scalar-free so-

⁶ This holds also the $n = 0$ limit of the GB model; there one finds $\alpha/M^2 = 0.3631$.

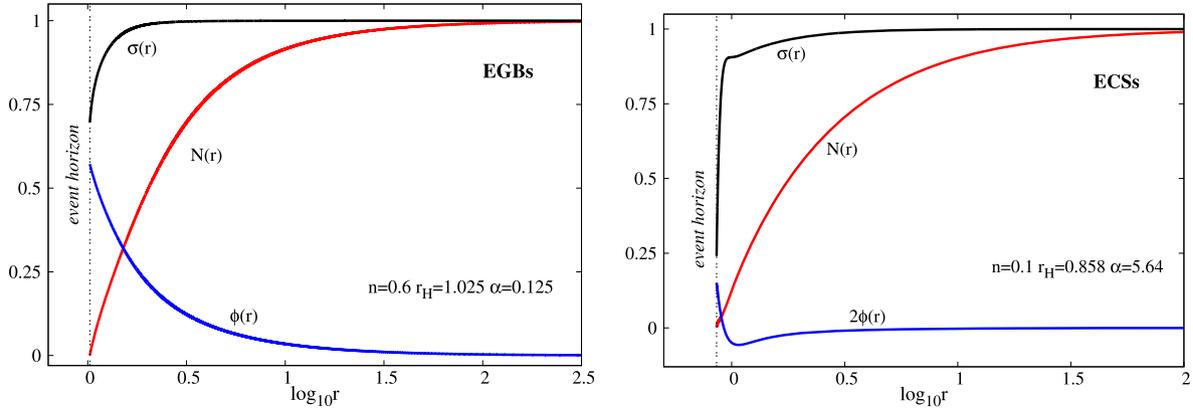


Fig. 3. Profile functions of a typical EGBs scalarised nodeless solution (left panel) and of a typical ECSs scalarised $k = 1$ solution (right panel).

lution (2.13): $N(r) \rightarrow 1$, $g(r) \rightarrow r^2$, $\sigma(r) \rightarrow 1$ and $\phi(r) \rightarrow 0$, as $r \rightarrow \infty$. The solution will also possess a horizon at $r = r_h > 0$, where $N(r_h) = 0$, and $g(r)$, $\sigma(r)$ are strictly positive.

Since the scalarised configurations are smoothly connected with the vacuum NUT solution, it is natural to choose the same metric gauge (2.12).⁷ Then we are left with a system of three non-linear ODEs (plus a constraint) for the functions N , σ and ϕ . These equations do not appear to possess closed form solutions, and are constructed by solving numerically a boundary value problem in the domain $r_h \leq r < \infty$ (with $r_h > 0$). As $r \rightarrow r_h$, one takes a power series expansion of the solution in $(r - r_h)$, the first terms being

$$\begin{aligned} N(r) &= N_1(r - r_h) + N_2(r - r_h)^2 + \dots, \\ \sigma(r) &= \sigma_0 + \sigma_1(r - r_h) + \dots, \\ \phi(r) &= \phi_0 + \phi_1(r - r_h) + \dots, \end{aligned} \quad (3.26)$$

with all coefficients in this expansion being fixed by the set $\{N_1, \sigma_0, \phi_0\}$. These parameters are subject to the following constraint:

$$N_1 r_h = 1 + \frac{96\alpha^2 N_1^2 (r_h^2 - n^2 \sigma_0^2) \phi_0^2}{(n^2 + r_h^2)^2}, \quad \text{for EGBs}, \quad (3.27)$$

$$N_1 r_h = 1 + \frac{192\alpha^2 N_1^3 n^2 r_h \sigma_0^2 \phi_0^2}{(n^2 + r_h^2)^2}, \quad \text{for ECSs}. \quad (3.28)$$

The leading order expansion in the far field is the same in both cases:

$$\begin{aligned} N(r) &= 1 - \frac{2M}{r} - \frac{Q_s^2 - 8n^2}{4r^2} + \dots, & \sigma(r) &= 1 - \frac{Q_s^2}{8r^2} + \dots, \\ \phi(r) &= \frac{Q_s}{r} + \frac{MQ_s}{r^2} + \dots, \end{aligned} \quad (3.29)$$

introducing the parameters M and Q_s which are fixed by numerics; M is identified as the mass of solutions, cf. (2.17). The constant Q_s can be interpreted as the ‘scalar charge’, giving a quantitative measure of ‘hairiness’ of the solutions.

For the EGBs model, the numerical integration was carried out using a standard shooting method. In this approach, we evaluate the initial conditions at $r = r_h + 10^{-6}$ for global tolerance 10^{-14} , adjusting for shooting parameters and integrating towards $r \rightarrow \infty$. The ECSs solutions were found by using a professional software package [29]. This solver employs a collocation method

for boundary-value differential equations and a damped Newton method of quasi-linearisation. In both cases, the input parameters were $\{r_h, n; \alpha\}$. We also remark that the configurations reported in this work are regular on and outside the horizon, all curvature invariants being finite.

3.2.2. Numerical results

As an illustration of the numerical results obtained in this way, the profile of a typical solution is shown in Fig. 3 for a nodeless scalar field in EGBs model and a $k = 1$ ECSs configuration.⁸

Let us now take a closer look at the domain of existence of the scalarised Schwarzschild-NUT solutions, starting with the EGBs model. The corresponding domain of existence is shown in Fig. 4. The solutions form a band in the (α, n) (or (α, Q_s)) plane, bounded by two curves: (i) the existence line and (ii) a set of critical solutions explained below. In units of M , for each α above a minimal value, scalarised solutions bifurcate from the vacuum Schwarzschild-NUT solution with a particular value of n at the existence line. Then, for given (n, M) , the scalarized solutions exist only within some α range, $\alpha_{min} < \alpha < \alpha_{max}$, with the limiting values increasing with the ratio n/M . Each constant α -branch starts on the existence line and ends at a critical configuration where the numerics stop to converge. The critical configuration does not possess any special properties. As with other solutions in various $d = 4$ EGBs models, its existence can be traced to the horizon condition (3.27): the reality of N_1 implies that solutions with given (r_h, n) can only exist if the coupling constant α is smaller than a critical value.

A similar picture to what has just been described can be found for the ECSs case. As can be anticipated from Fig. 1, however, there is a new ingredient: the existence of a double band structure of solutions – see Fig. 5. One can see that, for any n , no scalarized solutions exist below a critical value of α . Also, the $n \rightarrow 0$ limit cannot be reached for any finite α , as expected from the fact that the Chern–Simons term vanishes for $n = 0$. The domain of existence of the solutions is bounded again by the existence line and a critical line. The presence of critical ECSs configurations can be traced back, again, to the near horizon condition (3.28) which, for given (r_h, n) , stopped being satisfied for large enough α .

4. Further remarks

The main purpose of this work was to investigate the basic properties of the scalarised Schwarzschild-NUT solution, viewed as

⁷ Most of the ECSs solutions, however, were constructed for a metric gauge choice with $\sigma(r) = 1$ and unknown metric functions N, g .

⁸ For a given k , the typical profiles are similar in both models.

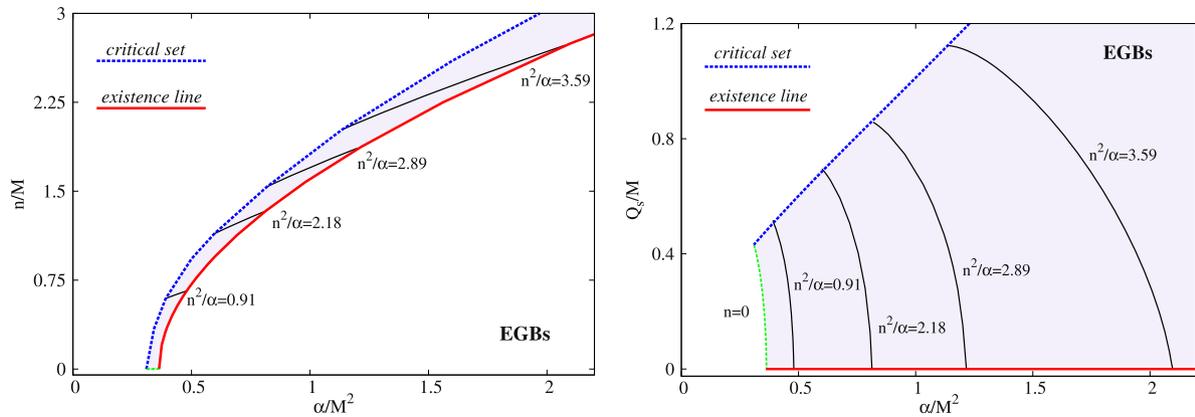


Fig. 4. Part of the domain of existence in the (α, n) and (α, Q_s) spaces (in units set by M) where scalarized Schwarzschild-NUT solutions exist in EGBs model. Only the fundamental ($k=0$) scalarization region is exhibited, but an infinite number of domains, indexed by the node number of the scalar field, should exist.

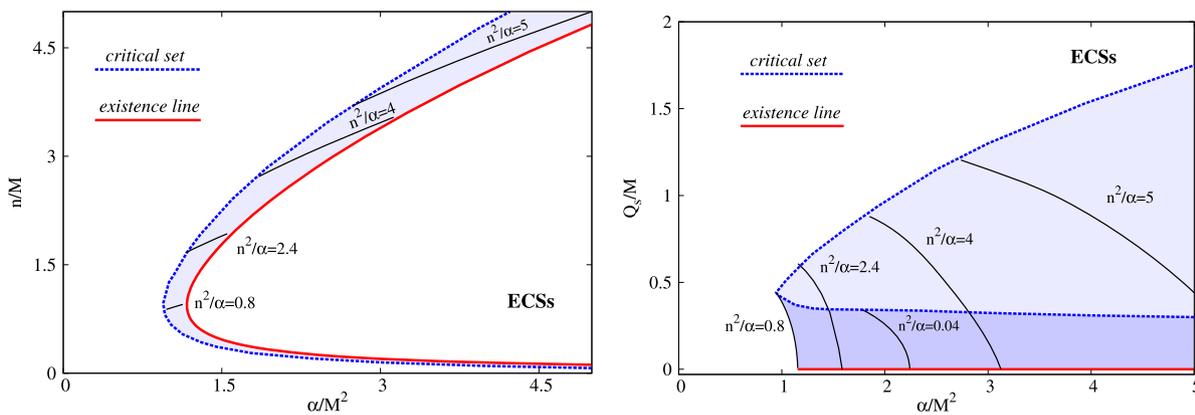


Fig. 5. Same as Fig. 4 for scalarised Schwarzschild-NUT solutions of the ECSs model.

a toy model for the scalarised rotating Kerr BH. Indeed, the line-element of the former solution is Kerr-like, in the sense that it has a crossed metric component $g_{\phi t}$, cf. (2.11). This term does not produce an ergoregion but it leads to an effect similar to the dragging of inertial frames [30]. In fact one can interpret the Schwarzschild-NUT spacetime as consisting of two counter-rotating regions, with a vanishing total angular momentum [31,32].

Our results show the following main trends. Firstly, for the EGBs model, but not for the ECSs case, scalarised solutions exist for the vanishing NUT charge case, $n=0$. This is easy to understand from the fact that only for $n \neq 0$, the Pontryagin density is excited in the scalar-free case, cf. eq. (2.16). This basic difference leads to a distinct domain of existence of solutions in the CS case, in particular exhibiting a double branch structure. In both cases the domain of existence of scalarised solution, for fixed NUT charge, is limited to a finite interval of coupling parameters α . At the lowest coupling, the limiting solutions define the existence line, i.e. the line of scalar-free solutions that can support a scalar cloud, as a test field configuration. At the largest coupling, the limiting solutions define a critical line of maximally scalarised solutions. This interval of couplings wherein scalarised solutions occur varies with the NUT charge as follows. In the GB case, increasing the NUT charge (for fixed M) demands a stronger non-minimal coupling for scalarisation; in the CS case, due to the double branch structure, both this and the opposite trend are found. We expect the same trends to occur, in terms of the horizon area rather than NUT charge, in the Kerr case, which is confirmed by preliminary results.

Albeit a rather unusual spacetime, the Schwarzschild-NUT solution has been generalised in various directions. In particular,

gauge fields have been added [33–35].⁹ The simplest of these solutions [33] represents an electrically charged version of the Schwarzschild-NUT solution, the Reissner–Nordström-NUT solution. This spacetime can be scalarised in a different way: by a non-minimal coupling of the scalar field to the Maxwell invariant, following the recent work [19]. That is, we take the action (2.1) but with

$$\mathcal{I} = F_{ab}F^{ab}, \quad (4.30)$$

where $F = dA$. The Reissner–Nordström-NUT solution can be expressed in the generic form (2.11), with (we consider the electric version of the solution only):

$$g = r^2 + n^2, \quad \sigma(r) = 1, \quad N(r) = \frac{r^2 - 2Mr - n^2 + Q^2}{r^2 + n^2} \quad (4.31)$$

and a $U(1)$ potential

$$A = \frac{Qr}{r^2 + n^2} (dt + 2n \cos \theta d\varphi). \quad (4.32)$$

Here Q is interpreted as an electric charge. Choosing a coupling function [19]

$$f(\phi) = e^{-\alpha\phi^2}, \quad (4.33)$$

⁹ Another generalisation, within a low-energy effective field theory from string theory is found in [36]. All these solutions share the same properties of the vacuum NUT metric, in particular the same asymptotic and causal structure.

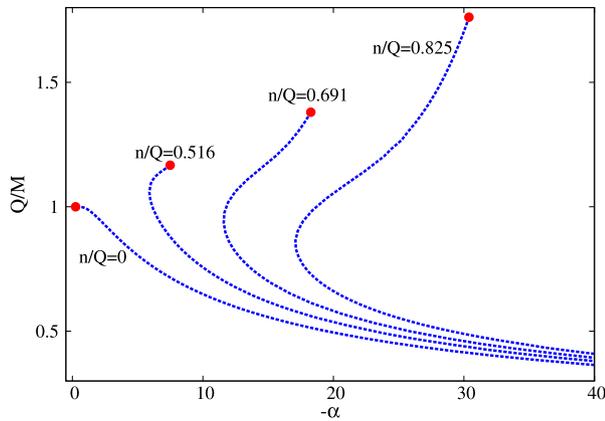


Fig. 6. $k = 0$ existence lines for the Reissner–Nordström–NUT solution for different values of the NUT charge normalised to the electric charge Q .

we have performed a preliminary study of the corresponding scalar clouds. The existence lines are shown in Fig. 6 for several values of the ratio n/M . The red dots at the end of the curves indicate limiting configurations where the function $N(r)$ in (4.31) develops a double zero (i.e. the extremal Reissner–Nordström BH for $n = 0$). The new feature for $n \neq 0$ is the existence, for the same value of the coupling constant, of two different unstable Reissner–Nordström–NUT configurations, distinguished by the value of the ratio Q/M .

Finally, we mention two other possible directions to bring together scalarisation and the Schwarzschild–NUT solution. Firstly, one could study the properties of the Euclideanised Schwarzschild–NUT solution, within the model (2.1). Could there be scalarised instantons? In this context, we recall that the Euclideanised Schwarzschild–NUT solution has found interesting applications in the context of quantum gravity [25]. Secondly, continuing the Schwarzschild–NUT metric through its horizon at $r = r_h$ leads to the Taub universe, a homogeneous, non-isotropic cosmology with an S^3 spatial topology. Whereas the Schwarzschild solution has a curvature singularity at $r = 0$, this is not the case for $n \neq 0$ and the radial coordinate in the Schwarzschild–NUT solution may span the whole real axis. Thus, following [37,38], it could be interesting to investigate the behaviour of the scalarised solutions inside the horizon.

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